Lecture 2: Modular Representations and Brauer Characters

Goal: Understand representations over fields of positive characteristic, define Brauer characters, and examine how character theory behaves when the characteristic of the field divides the group order.

1. Modular Representation Theory

Definition 2.1 (Modular System). Let G be a finite group and p a prime dividing $|G|$. A *modular system* is a triple $(K, \mathcal{O}, k)$ where:

- $\mathcal{O}$ is a complete discrete valuation ring (DVR),
- $K = \mathrm{Frac}(\mathcal{O})$ is a field of characteristic 0,
- $k = \mathcal{O}/\mathfrak{m}$ is the residue field of characteristic p.

Example 2.2. Let $\mathcal{O} = \mathbb{Z}_p$, then $K = \mathbb{Q}_p$, and $k = \mathbb{F}_p$.

Definition 2.3 (Modular Representation). A *modular representation* of G is a representation over a field F of characteristic p, where $p \mid |G|$.

Key Phenomenon: When $\mathrm{char}(F) \mid |G|$, Maschke's theorem fails, and not all representations are semisimple.

2. p-Singular and p-Regular Elements

Definition 2.4. Let p be a prime. An element $g \in G$ is called:

- *p-regular* if its order is not divisible by p,
- *p-singular* if its order is divisible by p.

3. Brauer Characters

Definition 2.5 (Brauer Character). Let $\rho : G \to \mathrm{GL}(V)$ be a modular representation over $\overline{\mathbb{F}}_p$. A *Brauer character* φ is a function:

$$\varphi(g) = \mathrm{Tr}(\tilde{\rho}(g))$$

defined only on p-regular elements, where $\tilde{\rho}$ is a lifting of ρ from characteristic 0 followed by reduction modulo p.

Theorem 2.6 (Brauer–Nesbitt). The Brauer character of a semisimple representation over a field of characteristic p is well-defined and determines the representation up to isomorphism.

4. Reduction Modulo p

Definition 2.7 (Reduction of Representations). Let $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ be a representation over the ring $\mathcal{O} \subset K$. Reducing ρ modulo $\mathfrak{m}$ gives a representation over k:

$$\overline{\rho} : G \to \mathrm{GL}_n(k).$$

Example 2.8. Take $G = S_3$, and let ρ be a two-dimensional irreducible complex representation. Reduce modulo 2 to obtain a representation over $\mathbb{F}_2$. The resulting module may become reducible or remain irreducible.

5. Decomposition of Ordinary Characters

Definition 2.9 (Decomposition Matrix). Let $\{\chi_i\}$ be ordinary irreducible characters and $\{\varphi_j\}$ the irreducible Brauer characters. The *decomposition matrix* $D = (d_{ij})$ records how ordinary characters restrict to p-regular elements:

$$\chi_i|_{p\text{-reg}} = \sum_j d_{ij}\varphi_j.$$

The decomposition matrix is usually a nonnegative integer matrix, not necessarily square, and encodes how ordinary representations split under reduction modulo p.

Proposition 2.10. The number of irreducible Brauer characters equals the number of p-regular conjugacy classes.

6. Examples

Example 2.11 (Decomposition Matrix of S_3, $p = 3$):

	χ_1	χ_2	χ_3
φ_1	1	0	1
φ_2	0	1	1

This matrix shows how the ordinary characters decompose over $\mathbb{F}_3$.

Example 2.12 (Kernel Phenomenon). When reducing characters, certain linear combinations vanish on p-regular classes. These generate the kernel of the projection from ordinary to Brauer character space.

7. Counterexamples

Counterexample 2.13. In characteristic 0, irreducible characters are orthonormal. In characteristic p, they are not. The inner product structure of modular characters is more delicate.

8. Summary

This lecture introduces the key objects of modular character theory:

- Modular representations differ from classical ones due to failure of semisimplicity.
- Brauer characters are only defined on p-regular elements.
- Decomposition matrices bridge classical and modular theory.

Coming Up in Lecture 3: We'll study *modular irreducibility criteria*, introduce the *Frobenius maps*, and connect representation traces to field structure. This will help explain why some modular characters live in extensions like $\mathbb{F}_4$, as seen in earlier lectures.